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Generalized Hausdorff and Weighted Mean Matrices as Operators on l_p

DAVID BORWEIN AND XIAOPENG GAO*

Department of Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7

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Two theorems are proved. Theorem 1 establishes sufficient conditions for a generalized Hausdorff matrix $H(\lambda, \alpha)$ either to be in $B(l_p)$ or not to be in $B(l_p)$. Theorem 2 shows, inter alia, that if $1 \leqslant p < \infty$, $a_n > 0$, $A_n := a_0 + a_1 + \cdots + a_n$, and $A_n/na_n \to c > 0$, then the weighted mean matrix M_a with weights a_n is in $B(l_p)$ if and only if $c < p$. There are two examples about cases when the conditions of the theorems are not satisfied. A short proof of the fact that weighted mean matrices are special generalized Hausdorff matrices is also given.

1. INTRODUCTION

Suppose throughout that $1 \leqslant p < \infty$, and that $A := (a_{nk})$ is a triangular matrix of complex numbers, that is $a_{nk} = 0$ for $n > k$. Let l_p be the Banach space of all complex sequences $x = \{x_n\}$ with norm

$$\|x\|_p := \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} < \infty,$$

and let $B(l_p)$ be the Banach algebra of all bounded linear operators on l_p. Thus $A \in B(l_p)$ if and only if $Ax \in l_p$ whenever $x \in l_p$, Ax being the sequence with nth term $(Ax)_n := \sum_{k=0}^{n} a_{nk} x_k$. Let

$$\|A\|_p := \sup_{\|x\|_p \leqslant 1} \|Ax\|_p,$$

so that $A \in B(l_p)$ if and only if $\|A\|_p < \infty$, in which case $\|A\|_p$ is the norm of A.

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Generalized Hausdorff Matrices

Suppose that $\lambda := \{\lambda_n\}$ is a sequence of real numbers with $\lambda_0 \geqslant 0$, $\lambda_n > 0$ for $n \geqslant 1$, and that α is a function of bounded variation on $[0, 1]$. For $0 \leqslant k \leqslant n$, let

$$\begin{aligned} \lambda_{nk}(t) &:= -\lambda_{k+1} \cdots \lambda_n \frac{1}{2\pi i} \int_C \frac{t^z \, dz}{(\lambda_k - z) \cdots (\lambda_n - z)}, \qquad 0 < t \leqslant 1, \\ \lambda_{nk}(0) &:= \lambda_{nk}(0+), \end{aligned} \tag{1}$$

$C = C_{nk}$ being a positively sensed Jordan contour enclosing $\lambda_k, \lambda_{k+1}, ..., \lambda_n$. Here and elsewhere we observe the convention that empty products, like $\lambda_{k+1} \cdots \lambda_n$ when $k = n$, have the value 1. Let

$$\lambda_{nk} := \int_0^1 \lambda_{nk}(t) \, d\alpha(t) \qquad \text{for} \quad 0 \leqslant k \leqslant n, \qquad \lambda_{nk} := 0 \qquad \text{for} \quad k > n,$$

and denote the triangular matrix (λ_{nk}) by $H(\lambda, \alpha)$. This is called a generalized Hausdorff matrix.

Weighted Mean Matrices

Let $a := \{a_n\}$ be a sequence of positive numbers and let $A_n := \sum_{k=0}^n a_k$. The weighted mean matrix $M_a := (a_{nk})$ is defined by setting

$$a_{nk} := \frac{a_k}{A_n} \qquad \text{for} \quad 0 \leqslant k \leqslant n, \qquad a_{nk} := 0 \qquad \text{for} \quad k > n.$$

Let

$$\begin{aligned} D_0 &:= (1 + \lambda_0) \, d_0 = 1, \\ D_n &:= \left(1 + \frac{1}{\lambda_1}\right) \cdots \left(1 + \frac{1}{\lambda_n}\right) \\ &=: (1 + \lambda_n) \, d_n \qquad \text{for} \quad n \geqslant 1. \end{aligned} \tag{2}$$

Then

$$D_n = \lambda_{n+1} d_{n+1} = \frac{\lambda_0}{1 + \lambda_0} + \sum_{k=0}^{n} d_k \qquad \text{for} \quad n \geqslant 0.$$

Also, it was proved in [5] that when all the λ_n's are distinct

$$\int_0^1 \lambda_{nk}(t) \, dt = \frac{d_k}{D_n} \qquad \text{for} \quad 0 \leqslant k \leqslant n. \tag{3}$$

Although a continuity argument shows that this is also true for more general λ_n, Lemma 1 (below) affords a shorter and more direct proof. When $\alpha(t) := t$ and $\lambda_0 := 0$, $H(\lambda, \alpha)$ reduces to the weighted mean matrix M_d with $d := \{d_n\}$ given by (2). Conversely if $d := \{d_n\}$ is a sequence of positive numbers with $d_0 := 1$, then (2) yields a sequence $\lambda := \{\lambda_n\}$ such that $H(\lambda, \alpha)$ becomes M_d when $\alpha(t) := t$.

The following theorem is due to Cass and Kratz [4].

THEOREM A. *Suppose that* $a_n = f(n)$ *where* $f(x)$ *is a logarithmico-exponential function for* $x > x_0$, *and that* $A_n/na_n \to c$. *Suppose also that* $p > 1$ *and* $(1/p) + (1/q) = 1$. *Then* $M_a \in B(l_p)$ *if and only if* $c < p$, *in which case*

$$\frac{p}{p-c} \leqslant \|M_a\|_p \leqslant \sigma_1^{1/q} \sigma_2^{1/p} < \infty$$

where

$$\sigma_1 = \sup_{n \geqslant 0} \sum_{k=0}^{n} \frac{a_k}{A_n} \left(\frac{n+1}{k+1}\right)^{1/p} \quad \text{and} \quad \sigma_2 = \sup_{k \geqslant 0} \sum_{n=k}^{\infty} \frac{a_k}{A_n} \left(\frac{k+1}{n+1}\right)^{1/q}.$$

Cass and Kratz showed that A_n/na_n necessarily tends to a finite or infinite limit when a_n is generated by a logarithmico-exponential function. Borwein [1] proved the following theorem.

THEOREM B. *If* $p \geqslant 1$, $c > 0$ *and*

$$\mu := \sup_{0 \leqslant k \leqslant n} \frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + c) \cdots (\lambda_{n-1} + c)} < \infty, \tag{4}$$

and if $\int_0^1 t^{-c/p} |d\alpha(t)| < \infty$, *then* $H(\lambda, \alpha) \in B(l_p)$, *and*

$$\|H(\lambda, \alpha)\|_p \leqslant \mu^{1/p} \int_0^1 t^{-c/p} |d\alpha(t)|.$$

Note. Although the proof of Theorem 1 in [1] in fact establishes Theorem B, the statement of Theorem 1 has the simpler condition

$$\lambda_{n+1} \leqslant c + \lambda_n \qquad \text{for} \quad n \geqslant n_0 \tag{5}$$

in place of (4). Evidently (5) implies (4), but as illustrated in Example 1 (below) it is possible for (4) to hold for some c and (5) to fail to hold for any c.

One of the objectives of this paper is to show that, subject to the existence of $c := \lim A_n/na_n$, the condition in Theorem A that a_n be generated by a logarithmico-exponential function is redundant when $c > 0$, and can be replaced by the far less restrictive condition that $\{a_n\}$ be eventually monotonic when $c = 0$. This objective is achieved by means of Theorem 2 (below) which is largely a specialization of our main result, Theorem 1 (below). In view of (2) and (3), the existence of $\lim A_n/na_n$ in Theorem 2 corresponds to the existence of $\lim \lambda_n/n$ in Theorem 1.

We shall prove the following two theorems:

THEOREM 1. *Suppose* $p \geqslant 1$. *Let* $c_1 := \liminf \lambda_n/n$ *and* $c_2 := \limsup \lambda_n/n$.

(i) *If* $\infty > c_1 > 0$, $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$, *and* α *is a non-decreasing function on* $[0, 1]$ *such that* $\alpha(0+) = \alpha(0)$, *then*

$$\|H(\lambda, \alpha)\|_p \geqslant \int_0^1 t^{-c_1/p}\, d\alpha(t).$$

In particular, if $\int_0^1 t^{-c_1/p}\, d\alpha(t) = \infty$, *then* $H(\lambda, \alpha) \notin B(l_p)$.

(ii) *If* $\lim \lambda_n/n = \infty$ (*i.e.*, $c_1 = \infty$), $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$, *and* α *is a non-decreasing function on* $[0, 1]$ *such that* $\alpha(0+) = \alpha(0) < \alpha(r)$ *for some* $r \in (0, 1)$, *then* $H(\lambda, \alpha) \notin B(l_p)$.

(iii) *If* $\int_0^1 t^{-c/p}\, |d\alpha(t)| < \infty$ *for some* $c > c_2$, *and* $c_1 > 0$, *then* $H(\lambda, \alpha) \in B(l_p)$, *and*

$$\|H(\lambda, \alpha)\|_p \leqslant \mu^{1/p} \int_0^1 t^{-c/p}\, |d\alpha(t)| < \infty$$

where μ *is given by* (4).

(iv) *If* $\lim \lambda_n/n = 0$ and $\lim(\lambda_{n+1} - \lambda_n)$ *exists, and if* $\int_0^1 t^{-\varepsilon}\, |d\alpha(t)| < \infty$ *for some* $\varepsilon > 0$, *then* $H(\lambda, \alpha) \in B(l_p)$ *for all* $p \geqslant 1$.

(v) *If the sequence* $\{d_n\}$ *given by* (2) *is eventually non-decreasing, and* $\int_0^1 t^{-1/p}\, |d\alpha(t)| < \infty$ *for some* $p \geqslant 1$, *then* $H(\lambda, \alpha) \in B(l_p)$.

THEOREM 2. *Suppose* $p \geqslant 1$. *Let* $c_1 := \liminf A_n/na_n$ *and* $c_2 := \limsup A_n/na_n$.

(i) *If either* $\sum a_n$ *is convergent, or* $\infty \geqslant c_1 \geqslant p$, *then* $M_a \notin B(l_p)$.

(ii) *If* $0 < c_1 \leqslant c_2 < p$, *then* $M_a \in B(l_p)$ *and*

$$\frac{p}{p - c_1} \leqslant \|M_a\|_p \leqslant \mu^{1/p} \frac{p}{p - c_2} < \infty$$

where μ *is given by* (4) *with* $\lambda_n := A_n/a_n - 1$ *and any* $c \in (c_2, p)$. *Furthermore, if* $0 < \lim A_n n a_n = c < p$ *and*

$$\frac{A_{n+1}}{a_{n+1}} \leqslant c + \frac{A_n}{a_n} \quad \text{for} \quad n \geqslant 0, \tag{6}$$

then $\|M_a\|_p = p/(p-c)$.

(iii) *If* $\lim A_n/na_n = 0$, *and*

$$\frac{A_{n+1}}{a_{n+1}} - \frac{A_n}{a_n}$$

tends to a limit, then $M_a \in B(l_p)$ *for all* $p \geqslant 1$.

(iv) *If* $\lim A_n/na_n = 0$, *and* $\{a_n\}$ *is eventually monotonic, then* $M_a \in B(l_p)$ *for all* $p > 1$.

2. Preliminary Results

Lemma 1. *If* D_n *and* d_n *satisfy* (2), *then* (3) *holds.*

Proof. Let Γ be a circle enclosing $\lambda_k, ..., \lambda_n$ and lying in the half-plane $\operatorname{Re} z > -\delta$ where $0 < \delta < 1$. For $0 < t \leqslant 1$ and $z \in \Gamma$, we have

$$\left| \frac{t^z}{(\lambda_k - z) \cdots (\lambda_n - z)} \right| \leqslant M t^{-\delta}$$

for some positive M independent of t and z. Hence, by Fubini's theorem,

$$\begin{aligned} \int_0^1 \lambda_{nk}(t)\, dt &= -\frac{\lambda_{k+1} \cdots \lambda_n}{2\pi i} \int_0^1 dt \int_\Gamma \frac{t^z\, dz}{(\lambda_k - z) \cdots (\lambda_n - z)} \\ &= -\frac{\lambda_{k+1} \cdots \lambda_n}{2\pi i} \int_\Gamma \frac{dz}{(z+1)(\lambda_k - z) \cdots (\lambda_n - z)}. \end{aligned}$$

Let

$$f(z) := \frac{1}{(z+1)(\lambda_k - z) \cdots (\lambda_n - z)}.$$

Then $\int_{|z|=m} f(z)\, dz \to 0$ as $m \to \infty$, and so

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = -\operatorname{Res}(f(z), -1) = -\frac{1}{(\lambda_k + 1) \cdots (\lambda_n + 1)}.$$

Consequently

$$\int_0^1 \lambda_{nk}(t)\,dt = \frac{\lambda_{k+1}\cdots\lambda_n}{(\lambda_k+1)\cdots(\lambda_n+1)} = \frac{d_k}{D_n}. \quad \blacksquare$$

The following lemma is included here for convenience. Its proof is given in [2].

LEMMA 2. *Under the hypotheses in the definition of a generalized Hausdorff matrix, we always have*

$$0 \leqslant \lambda_{nj}(t) \leqslant \sum_{k=0}^{n} \lambda_{nk}(t) \leqslant 1 \qquad \text{for} \quad 0 \leqslant t \leqslant 1, \quad 0 \leqslant j \leqslant n.$$

If, in addition, $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$, *then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \lambda_{nk}(t) = \begin{cases} 0 & \text{if} \quad t=0 \quad \text{and} \quad \lambda_0 > 0 \\ 1 & \text{otherwise.} \end{cases}$$

The next lemma is essentially Theorem 4 of [3] with superficial changes to make it a little more general and also easier to apply. We supply a proof here for completeness.

LEMMA 3. *Suppose that* $a_{nk} \geqslant 0$ *for* $0 \leqslant k \leqslant n$, $a_{nk} = 0$ *for* $k > n$, *and that* $\{b_n\}$ *is a bounded sequence of positive numbers such that* $\sum b_n = \infty$. *Let*

$$\sigma_n := \sum_{k=0}^{n} a_{nk} \left(\frac{b_k}{b_n}\right)^{1/p}.$$

If $\sigma := \liminf \sigma_n$ *and* $A := (a_{nk})$, *then* $\|A\|_p \geqslant \sigma$. *In particular, if* $\sigma = \infty$, *then* $A \notin B(l_p)$.

Proof. Suppose without loss in generality that $\sigma > 0$, and let $0 < \mu < \lambda < \sigma$. Let

$$T_n := \prod_{k=0}^{n} \left(1 - \frac{b_k}{b}\right)^{-1} \qquad \text{where} \quad b > \sup_{k \geqslant 0} b_k.$$

Then $t_n := T_n - T_{n-1} = T_n b_n / b$ for $n \geqslant 1$, and $T_n = T_0 + t_1 + \cdots + t_n \to \infty$. Let

$$y_n := \sum_{k=0}^{n} a_{nk} x_k \qquad \text{where} \quad x_k := \left(\frac{b_k}{T_k^{\varepsilon}}\right)^{1/p}, \qquad \varepsilon > 0.$$

By Dini's theorem, $\{x_k\} \in l_p$. Further, there is a positive integer N independent of ε such that for $n \geqslant N$

$$y_n = x_n \sum_{k=0}^{n} a_{nk} \left(\frac{b_k}{b_n}\right)^{1/p} \left(\frac{T_n}{T_k}\right)^{\varepsilon/p}$$
$$\geqslant x_n \sum_{k=0}^{n} a_{nk} \left(\frac{b_k}{b_n}\right)^{1/p} \geqslant \lambda x_n.$$

Now choose ε so small that

$$\sum_{n=N}^{\infty} x_n^p = b \sum_{n=N}^{\infty} \frac{t_n}{T_n^{1+\varepsilon}} \geqslant \left(\frac{\mu}{\lambda}\right)^p \sum_{n=0}^{\infty} x_n^p.$$

Then

$$\sum_{n=0}^{\infty} y_n^p \geqslant \lambda^p \sum_{n=N}^{\infty} x_n^p \geqslant \mu^p \sum_{n=0}^{\infty} x_n^p.$$

Therefore $\|A\|_p \geqslant \mu$ and, since μ is an arbitrary number in the interval $(0, \sigma)$, it follows that $\|A\|_p \geqslant \sigma$. ∎

3. Proofs of the Main Results

To prove part (ii) of Theorem 1 we need the following lemma.

Lemma 4. *If* $\liminf \lambda_n/n > 0$, *and* $c_2 := \limsup \lambda_n/n < c < \infty$, *then*

$$\mu := \sup_{0 \leqslant k \leqslant n} \frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + c) \cdots (\lambda_{n-1} + c)} < \infty.$$

Proof. Let n_0 be a positive integer such that $\lambda_n/n \leqslant c$ when $n \geqslant n_0$. *For* $n \geqslant k > n_0$ *we have*

$$\frac{\lambda_k + c}{\lambda_k} \cdots \frac{\lambda_{n-1} + c}{\lambda_{n-1}} \geqslant \left(1 + \frac{1}{k}\right) \cdots \left(1 + \frac{1}{n-1}\right) = \frac{n}{k},$$

so that

$$\frac{\lambda_{k+1}}{\lambda_k + c} \cdots \frac{\lambda_n}{\lambda_{n-1} + c} \leqslant \frac{k}{\lambda_k} \frac{\lambda_n}{n} \leqslant \left(\sup_{k \geqslant n_0} \frac{k}{\lambda_k}\right) c =: M_1 < \infty$$

Let

$$M_2 := \sup_{0 \leqslant k \leqslant n \leqslant n_0} \frac{\lambda_{k+1}}{\lambda_k + c} \cdots \frac{\lambda_n}{\lambda_{n-1} + c}.$$

Then for $0 \leqslant k \leqslant n_0 < n$ we have

$$\frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + c) \cdots (\lambda_{n-1} + c)} = \frac{\lambda_{k+1} \cdots \lambda_{n_0}}{(\lambda_k + c) \cdots (\lambda_{n_0-1} + c)} \cdot \frac{\lambda_{n_0+1} \cdots \lambda_n}{(\lambda_{n_0} + c) \cdots (\lambda_{n-1} + c)} \leqslant M_1 M_2 < \infty,$$

and so $\mu \leqslant \max(M_1, M_2, M_1 M_2) < \infty$. ∎

Proof of Theorem 1. (i) Let $0 < w < c_1/p$, let

$$b_n := \left(\frac{\lambda_1 \cdots \lambda_n}{(\lambda_1 + w) \cdots (\lambda_n + w)} \right)^p, \qquad \tilde{\lambda}_n := \lambda_n + w,$$

and define $\tilde{\lambda}_{nk}(t)$ by (1) with $\{\tilde{\lambda}_n\}$ in place of $\{\lambda_n\}$. Since $wp < c_1$ there is a positive integer n_0 such that $\lambda_n \geqslant nwp$ for all $n \geqslant n_0$. Hence, for $n \geqslant n_0$,

$$\frac{b}{b_{n-1}} = \frac{\lambda_n^p}{(\lambda_n + w)^p} = \left(1 + \frac{w}{\lambda_n}\right)^{-p} \geqslant 1 - \frac{pw}{\lambda_n} \geqslant 1 - \frac{1}{n} = \frac{n-1}{n},$$

and so $b_n \geqslant b_{n_0} n_0 / n$. It follows that $\sum b_n = \infty$. Further, for $0 \leqslant k \leqslant n$, $0 < t \leqslant 1$,

$$\begin{aligned}
\lambda_{nk}(t) \left(\frac{b_k}{b_n}\right)^{1/p} &= -(\lambda_{k+1} + w) \cdots (\lambda_n + w) \frac{1}{2\pi i} \int_C \frac{t^z \, dz}{(\lambda_k - z) \cdots (\lambda_n - z)} \\
&= -\tilde{\lambda}_{k+1} \cdots \tilde{\lambda}_n \frac{t^{-w}}{2\pi i} \int_{C_1} \frac{t^{z_1} \, dz_1}{(\tilde{\lambda}_k - z_1) \cdots (\tilde{\lambda}_n - z_1)} \qquad (z_1 := z + w) \\
&= \tilde{\lambda}_{nk}(t)\, t^{-w}
\end{aligned}$$

By Lemma 2 and Fatou's theorem,

$$\liminf_{n \to \infty} \sum_{k=0}^{n} \lambda_{nk} \left(\frac{b_k}{b_n}\right)^{1/p} = \liminf_{n \to \infty} \sum_{k=0}^{n} \int_0^1 \tilde{\lambda}_{nk}(t)\, t^{-w} \, d\alpha(t) \geqslant \int_0^1 t^{-w} \, d\alpha(t),$$

and hence, by Lemma 3,

$$\|H(\lambda, \alpha)\|_p \geqslant \int_0^1 t^{-w} \, d\alpha(t). \tag{7}$$

It follows, on letting $w \to c_1/p$ from the left and appealing to the monotonic convergence theorem, that

$$\|H(\lambda, \alpha)\|_p \geqslant \int_0^1 t^{-c_1/p}\, d\alpha(t). \tag{8}$$

Notice that (8) is true when the right side is either finite or infinite. In the case when it is infinite, (8) means precisely that $H(\lambda, \alpha) \notin B(l_p)$.

(ii) For any $w > 0$, we obtain (7) exactly as in part (i). It follows that $\|H(\lambda, \alpha)\|_p \geqslant \int_0^r t^{-w}\, d\alpha(t) \geqslant r^{-w} \int_0^r d\alpha(t) \to \infty$ as $w \to \infty$. Thus $H(\lambda, \alpha) \notin B(l_p)$.

(iii) This is an immediate consequence of Lemma 4 and Theorem B.

(iv) Observe that

$$\frac{\lambda_n}{n} = \frac{1}{n} \sum_{k=0}^{n-1} (\lambda_{k+1} - \lambda_k) + \frac{\lambda_0}{n}.$$

So $\lim \lambda_n/n = 0$ and the existence of $\lim(\lambda_{n+1} - \lambda_n)$ imply that $\lim(\lambda_{n+1} - \lambda_n) = 0$. Hence $\lambda_{n+1} \leqslant \lambda_n + \varepsilon p$ for large n, and so, by Theorem B, $H(\lambda, \alpha) \in B(l_p)$.

(v) This also follows from Theorem B since, by (2),

$$\lambda_{n+1} - \lambda_n = \frac{D_{n+1}}{d_{n+1}} - \frac{D_n}{d_n} = D_n \left(\frac{1}{d_{n+1}} - \frac{1}{d_n} \right) + 1 \leqslant 1$$

for large n. ▮

Proof of Theorem 2. (i) If $\sum a_n < \infty$, then $\sum (a_0/A_n)^p = \infty$; but this implies that $M_a e^0 \notin l_p$, where $e^0 := (1, 0, 0, \ldots)$, so that $M_a \notin B(l_p)$. That $M_a \notin B(l_p)$ when $\infty \geqslant c_1 \geqslant p$ and $\sum a_n = \infty$, follows directly from parts (i) and (ii) of Theorem 1 with $\alpha(t) := t$ and $\lambda_n := A_n/a_n - 1$, since $A_n \to \infty$ if and only if $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$.

(ii) This follows from parts (i) and (iii) of Theorem 1, the final conclusion being justified because the appropriate $\mu = 1$ when (6) holds.

(iii) This follows from Theorem 1(iv) and Lemma 1, since

$$\frac{A_{n+1}}{a_{n+1}} - \frac{A_n}{a_n} = \lambda_{n+1} - \lambda_n,$$

and $\int_0^1 t^{-\varepsilon}\, dt < \infty$ when $\varepsilon < 1$.

(iv) First we prove that the hypothesis $\lim A_n/na_n = 0$ implies that $\lim a_n = \infty$. (In fact the hypothesis implies that $n^{-c}a_n \to \infty$ for every real constant c). Let $\alpha_n := na_n/A_n$. Then

$$\log \frac{A_n}{A_{n-1}} = -\log\left(1 - \frac{\alpha_n}{n}\right) > \frac{\alpha_n}{n},$$

so that $\log A_n > \log A_0 + \sum_{k=1}^{n} \alpha_k/k$. Since $\alpha_n \to \infty$, it follows that $\log A_n/\log n \to \infty$, and hence that, for any given c and sufficiently large n, $\log A_n > (c+1)\log n$, or $A_n > n^{c+1}$. Therefore $n^{-c}a_n = n^{-c-1}\alpha_n A_n > \alpha_n \to \infty$ as $n \to \infty$.

Hence, since $\{a_n\}$ is eventually monotonic it must be eventually non-decreasing, and so $M_a \in B(l_p)$ for every $p > 1$ by Theorem 1(v). ▌

4. Examples

In this section we deal with two examples. For the first we construct a generalized Hausdorf matrix $H(\lambda, \alpha) \in B(l_{p_0})$, with α increasing, for which $\int_0^1 t^{-c/p_0}\, d\alpha(t) = \infty$ for every $c > c_2$, where $p_0 \geqslant 1$ and c_2 is as in Theorem 1(iii). This will show that the conclusion of Theorem 1(iii) can hold when its main condition is not satisfied. The example will also show that (4) can hold with $c = 1$ while (5) can fail to hold for any c.

Example 1. Suppose that $p_0 > 1$. Let

$$\lambda_n := m^2 \qquad \text{for} \quad m^2 \leqslant n < (m+1)^2, \qquad m = 0, 1, 2, ...,$$

$$d\alpha(t) := \frac{t^{1/p_0}}{t \log^2 t/2}\, dt \qquad \text{for} \quad t \in (0, 1].$$

Observe that $c_2 := \lim \lambda_n/n = 1$, and $\int_0^1 t^{-c/p_0}\, d\alpha(t) = \infty$ for all $c > c_2$. Thus Theorem 1(ii) cannot be used to prove that $H(\lambda, \alpha) \in B(l_{p_0})$. Instead we shall appeal to Theorem B. For $m \geqslant 1$, we have that

$$\beta_m := \frac{\lambda_{m^2+1} \cdots \lambda_{(m+1)^2}}{(\lambda_{m^2}+1) \cdots (\lambda_{m^2+2m}+1)} = \left(\frac{m^2}{m^2+1}\right)^{2m+1} \left(\frac{m+1}{m}\right)^2.$$

Since

$$\left(\frac{m^2+1}{m^2}\right)^{m+1/2} = \left(1 + \frac{1}{m^2}\right)^{m+1/2} \geqslant 1 + \left(m + \frac{1}{2}\right)\frac{1}{m^2} > 1 + \frac{1}{m} = \frac{m+1}{m},$$

it follows that $\beta_m \leqslant 1$ for $m \geqslant 1$. Also because

$$\lim_{m \to \infty} \frac{\lambda_{m^2}}{\lambda_{m^2-1}+1} = 1,$$

and $\lambda_k/(\lambda_{k-1}+1) < 1$ when k is not a perfect square, we get

$$\sup_{0 \leqslant k \leqslant n} \frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k+1) \cdots (\lambda_{n-1}+1)} < \infty.$$

Since

$$\int_0^1 t^{-1/p_0}\, d\alpha(t) = \int_0^1 \frac{dt}{t \log^2 t/2} < \infty,$$

it follows that $H(\lambda, \alpha) \in B(l_{p_0})$ by Theorem B with $c = 1$. In fact this theorem shows that $H(\lambda, \alpha) \in B(l_p)$ for all $p \geqslant p_0$. On the other hand, a simple consequence of Theorem 1(i) is that $H(\lambda, \alpha) \notin B(l_p)$ for $1 < p < p_0$.

Finally, we see that (4) holds with $c = 1$, but that (5) cannot hold for any c since $\limsup(\lambda_{n+1} - \lambda_n) = \infty$.

The second example will show us that it is possible for $M_a \notin B(l_p)$ when $A_n/na_n \to 0$, although Theorem 2 tells us that $M_a \in B(l_p)$ when $A_n/na_n \to c \in (0, p)$. Thus the condition $A_n/na_n \to 0$ needs to be augmented, as in parts (iii) and (iv) of Theorem 2, in order to ensure that $M_a \in B(l_p)$. Correspondingly the condition $\lambda_n/n \to 0$ needs to be augmented, as in part (iv) of Theorem 1, in order to ensure that $H(\lambda, \alpha) \in B(l_p)$.

EXAMPLE 2. Define the weighted mean matrix M_a with $a := \{a_n\}$ as follows:

$$a_n := \begin{cases} 1 & \text{for} \quad n = 0 \\ 2^m & \text{for} \quad m^2 < n < (m+1)^2, \qquad m = 1, 2, ... \\ m2^{m+1} & \text{for} \quad n = m^2. \end{cases}$$

Then

$$a_{m^2} = \sum_{k=m^2+1}^{(m+1)^2-1} a_k,$$

and so the partial sum

$$A_{(m+1)^2-1} = 1 + \sum_{k=1}^{m} k2^{k+2} = (m-1)\, 2^{m+3} + 9.$$

Hence, for $m^2 \leqslant n < (m+1)^2$,

$$\frac{na_n}{A_n} \geqslant \frac{m^2 2^m}{(m-1)\,2^{m+3}+9} \to \infty,$$

and

$$\frac{a_{m^2}}{A_n} \geqslant \delta_m := \frac{m 2^{m+1}}{(m-1)\,2^{m+3}+9} \to \frac{1}{4}.$$

Now let $p > 1$, and define $x := \{x_k\} \in l_p$ by setting

$$x_k := \begin{cases} \dfrac{1}{m^{1/p} \log m} & \text{if } \; k = m^2, \quad m = 2, 3, ... \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$y_n := \frac{1}{A_n} \sum_{k=0}^{n} a_k x_k.$$

Then, for $4 \leqslant m^2 \leqslant n < (m+1)^2$,

$$y_n \geqslant \frac{a_{m^2}}{A_n} x_{m^2} \geqslant \frac{\delta_m}{m^{1/p} \log m}.$$

Thus

$$\sum_{n=m^2}^{(m+1)^2-1} y_n^p \geqslant \frac{2m\delta_m^p}{m \log^p m}, \qquad \text{and so} \qquad \sum_{n=0}^{\infty} y_n^p \geqslant \sum_{m=2}^{\infty} \frac{2\delta_m^p}{\log^p m} = \infty.$$

Consequently $M_a \notin B(l_p)$, even though $\lim A_n/na_n = 0$.

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